Viability Kernel for Ecosystem Management Models

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Abstract

We consider sustainable management issues formulated within the framework of control theory. The problem is one of controlling a discrete–time dynamical system (*e. g.* population model) in the presence of state and control constraints, representing conflicting economic and ecological issues for instance. The viability kernel is known to play a basic role for the analysis of such problems and the design of viable control feedbacks, but its computation is not an easy task in general. We study the viability of nonlinear generic ecosystem models under preservation and production constraints. Under simple conditions on the growth rates at the boundary constraints, we provide an explicit description of the viability kernel. A numerical illustration is given for the hake–anchovy couple in the Peruvian upwelling ecosystem.

Key words: control theory; state constraints; viability; predator–prey; ecosystem management; Peruvian upwelling ecosystem.

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Contents

1 Introduction

This paper deals with the control of discrete–time dynamical systems of the form $x(t+1) = f\big(x(t), u(t)\big)$, $t \in \mathbb{N}$, with state $x(t) \in \mathbb{X}$ and control $u(t) \in \mathbb{U}$, in the presence of state and control constraints $\big(x(t), u(t)\big) \in \mathbb{D}$. The subset $\mathbb{D} \subset \mathbb{X} \times \mathbb{U}$ describes "acceptable configurations of the system". Such problems of dynamic control under constraints refer to viability (Aubin, 1991) or invariance (Clarke, Ledayev, Stern, and Wolenski, 1995) frameworks. From the mathematical viewpoint, most of viability and weak invariance results are addressed in the continuous time case. However, some mathematical works deal with the discrete-time case. This includes the study of numerical schemes for the approximation of the viability problems of the continuous dynamics as in (Aubin, 1991; Saint-Pierre, 1994; Quincampoix and Saint-Pierre, 1995). In the control theory literature, problems of

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constrained control have also been addressed in the discrete time case (see the survey paper (Blanchini, 1999)); reachability of target sets or tubes for nonlinear discrete time dynamics is examined in (Bertsekas and Rhodes, 1971).

We consider sustainable management issues which can be formulated within such a framework as in (Béné and Doyen, 2000, 2003; Béné, Doyen, and Gabay, 2001; Eisenack, Sheffran, and Kropp, 2006; Martinet and Doyen, 2007; Mullon, Cury, and Shannon, 2004; Rapaport, Terreaux, and Doyen, 2006; De Lara, Doyen, Guilbaud, and Rochet, 2007; De Lara and Doyen, 2008).

The time index t is an integer and the time period $[t, t+1[$ may be a year, a month, etc. The dynamics is generally a population dynamics, with state vector $x(t)$ being either the biomass of a single species, or a couple of biomasses for a predator–prey system, or a vector of abundances at ages for one or for several species, or abundances at different spatial patches, etc. The control $u(t)$ may represent catches, harvesting mortality or harvesting effort. The set $\mathbb{D}$ may include biological, ecological and economic objectives as in (Béné, Doyen, and Gabay, 2001). For instance, if the state x is a vector of abundances at ages and the control u is a harvesting effort, $\mathbb{D} = \{(x, u) \mid B(x) \geq b^\flat, E(x, u) \geq e^\flat\}$ represents acceptable or acceptable configurations where conservation is ensured by an biological indicator $B(x) \geq b^\flat$ (spawning stock biomass above a reference point, for instance) and economics is taken into account via minimal catches $E(x, u) \geq e^\flat$ (catches $E(x, u)$ above a threshold).

The viability kernel $\mathbb{V}(f, \mathbb{D})$ associated with the dynamics f and the acceptable set $\mathbb{D}$ is known to play a basic role for the analysis of such problems and the design of viable control feedbacks. Unfortunately, its computation is not an easy task in general. In this paper, we focus on the case where the viability kernel is given by a finite algorithm, with emphasis on population dynamics ecosystem models. For nonlinear predator–prey systems, one can find descriptions of viability kernels for Lotka–Volterra systems in (Bonneuil and Müllers, 1997) and population viability analysis in three trophic-level food chains in (Bonneuil and Saint-Pierre, 2005). In these latter references, the control strategies are carried by the species ("fitness-maximizing" strategies), the constraint set corresponds to having high enough densities, and

the models are in continuous time. In our approach, the controls are the harvesting efforts, the constraint set includes both high abundances and high catches, and the model is in discrete time.

The paper is organized as follows. Section 2 is devoted to recalls on discrete–time viability and its possible use for sustainable management. For generic nonlinear ecosystem models, we provide an explicit description of the viability kernel for preservation and production constraints in Section 3, together with viable controls. An illustration in ecosystem management and numerical applications are given for the hake–anchovy couple in the Peruvian upwelling ecosystem in Section 4.

2   Discrete–time viability

Let us consider a nonlinear control system described in discrete–time by the difference equation

$$
\begin{cases}
x(t+1) = f\big(x(t), u(t)\big) \ \text{ for all } \ t \in \mathbb{N}, \\
x(0) = x_0 \ \text{ given},
\end{cases}
\tag{1}
$$

where the *state variable* $x(t)$ belongs to the finite dimensional state space $\mathbb{X} = \mathbb{R}^{n_\mathbb{X}}$, the *control variable* $u(t)$ is an element of the *control set* $\mathbb{U} = \mathbb{R}^{n_\mathbb{U}}$ while the *dynamics* f maps $\mathbb{X} \times \mathbb{U}$ into $\mathbb{X}$.

A controller or a decision maker describes "acceptable configurations of the system" through a set $\mathbb{D} \subset \mathbb{X} \times \mathbb{U}$ termed the *acceptable set*

$$
\big(x(t), u(t)\big) \in \mathbb{D} \ \text{ for all } \ t \in \mathbb{N} \,,
\tag{2}
$$

where $\mathbb{D}$ includes both system states and controls constraints.

The *state constraints set* $\mathbb{V}^0$ associated with $\mathbb{D}$ is obtained by projecting the acceptable set $\mathbb{D}$ onto the state space $\mathbb{X}$:

$$
\mathbb{V}^0 := \mathrm{Proj}_{\mathbb{X}}(\mathbb{D}) = \{x \in \mathbb{X} \mid \exists u \in \mathbb{U}, \ (x, u) \in \mathbb{D}\} \,.
\tag{3}
$$

Viability is defined as the ability to choose, at each time step $t \in \mathbb{N}$, a control $u(t) \in \mathbb{U}$ such that the system configuration remains acceptable. More precisely, the system is viable

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if the following feasible set is not empty:

$$\mathbb{V}(f, \mathbb{D}) := \left\{ x_0 \in \mathbb{X} \,\middle|\, \begin{array}{c} \exists \, (u(0), u(1), \ldots) \text{ and } (x(0), x(1), \ldots) \\ \text{satisfying (1) and (2)} \end{array} \right\}. \tag{4}$$

The set $\mathbb{V}(f, \mathbb{D})$ is called the *viability kernel* (Aubin, 1991) associated with the dynamics f and the acceptable set $\mathbb{D}$. By definition, we have $\mathbb{V}(f, \mathbb{D}) \subset \mathbb{V}^0 = \text{Proj}_\mathbb{X}(\mathbb{D})$ but, in general, the inclusion is strict. For a decision maker or control designer, knowing the viability kernel has practical interest since it describes the states from which controls can be found that maintain the system in an acceptable configuration forever. However, computing this kernel is not an easy task in general.

We now focus on the tools to achieve viability. A subset $\mathbb{V}$ is said to be *weakly invariant* for the dynamics f in the acceptable set $\mathbb{D}$, or a *viability domain* of f in $\mathbb{D}$, if

$$\forall x \in \mathbb{V}, \quad \exists u \in \mathbb{U}, \quad (x, u) \in \mathbb{D} \text{ and } f(x, u) \in \mathbb{V}. \tag{5}$$

That is, if one starts from $\mathbb{V}$, an acceptable control may transfer the state in $\mathbb{V}$.

Moreover, according to viability theory (Aubin, 1991), the viability kernel $\mathbb{V}(f, \mathbb{D})$ turns out to be the union of all viability domains, or also the largest viability domain:

$$\mathbb{V}(f, \mathbb{D}) = \bigcup \left\{ \mathbb{V}, \, \mathbb{V} \subset \mathbb{V}^0, \, \mathbb{V} \text{ viability domain for } f \text{ in } \mathbb{D} \right\}. \tag{6}$$

A major interest of such a property lies in the fact that any viability domain for the dynamics f in the acceptable set $\mathbb{D}$ provides a *lower approximation* of the viability kernel. An *upper approximation* $\mathbb{V}_k$ of the viability kernel is given by the so called *viability kernel until time k associated with f in $\mathbb{D}$*:

$$\mathbb{V}_k := \left\{ x_0 \in \mathbb{X} \,\middle|\, \begin{array}{l} \exists \, (u(0), u(1), \ldots, u(k)) \text{ and } (x(0), x(1), \ldots, x(k)) \\ \text{satisfying (1) for } t = 0, \ldots, k-1 \\ \text{and (2) for } t = 0, \ldots, k \end{array} \right\}. \tag{7}$$

We have

$$\mathbb{V}(f, \mathbb{D}) \subset \mathbb{V}_{k+1} \subset \mathbb{V}_k \subset \mathbb{V}_0 = \mathbb{V}^0 \text{ for all } k \in \mathbb{N}. \tag{8}$$

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It may be seen by induction that the decreasing sequence of viability kernels until time k satisfies

$$\mathbb{V}_0 = \mathbb{V}^0 \text{ and } \mathbb{V}_{k+1} = \{x \in \mathbb{V}_k \mid \exists u \in \mathbb{U}, (x, u) \in \mathbb{D} \text{ and } f(x, u) \in \mathbb{V}_k\} . \tag{9}$$

By (8), such an algorithm provides approximation from above of the viability kernel as follows:

$$\mathbb{V}(f, \mathbb{D}) \subset \bigcap_{k \in \mathbb{N}} \mathbb{V}_k = \lim_{k \to +\infty} \downarrow \mathbb{V}_k . \tag{10}$$

Conditions ensuring that equality holds may be found in (Saint-Pierre, 1994). Notice that, when the decreasing sequence $(\mathbb{V}_k)_{k \in \mathbb{N}}$ of viability kernels up to time k is stationary, its limit is the viability kernel. Indeed, if $\mathbb{V}_k = \mathbb{V}_{k+1}$ for some k, then $\mathbb{V}_k$ is a viability domain by (9). Now, by (4), $\mathbb{V}(f, \mathbb{D})$ is the largest of viability domains. As a consequence, $\mathbb{V}_k = \mathbb{V}(f, \mathbb{D})$ since $\mathbb{V}(f, \mathbb{D}) \subset \mathbb{V}_k$ by (8). We shall use this property in the following Sect. 3.

Once the viability kernel, or any approximation, or a viability domain is known, we have to consider the management or control issue, that is the problem of selecting suitable controls at each time step. For any viability domain $\mathbb{V}$ and any state $x \in \mathbb{V}$, the following subset $\mathbb{U}_{\mathbb{V}}(x)$ of the decision set $\mathbb{U}$ is not empty:

$$\mathbb{U}_{\mathbb{V}}(x) := \{u \in \mathbb{U} \mid (x, u) \in \mathbb{D} \text{ and } f(x, u) \in \mathbb{V}\} . \tag{11}$$

Therefore $\mathbb{U}_{\mathbb{V}(f, \mathbb{D})}(x)$ stands for the largest set of *viable controls associated with* $x \in \mathbb{X}$. Then, the decision design consists in the choice of a viable *feedback* control, namely any selection $\Psi : \mathbb{X} \to \mathbb{U}$ which associates with each state $x \in \mathbb{V}(f, \mathbb{D})$ a control $u = \Psi(x)$ satisfying $\Psi(x) \in \mathbb{U}_{\mathbb{V}(f, \mathbb{D})}(x)$.

In the context of sustainable management, viability concepts and methods may help giving a framework for setting decision making. First, one should delineate *perpetual objectives* (the set $\mathbb{D}$). Second, *operational objectives (advice)* can be obtained as viable controls. This approach is illustrated in (De Lara, Doyen, Guilbaud, and Rochet, 2006), and especially in (De Lara, Doyen, Guilbaud, and Rochet, 2007) for fishery management.

3 Viable control of generic nonlinear ecosystem models

For a generic ecosystem model, we provide an explicit description of the viability kernel. Then, we shall specify the results for predator–prey systems, in particular for discrete-time Lotka–Volterra models.

3.1 Dynamics and constraints for an ecosystem management model

For simplicity, we consider a two–dimensional state model. However, the following Proposition 1 may be easiliy extended to n–dimensional systems as long as each species is harvested by a specific device: one species, one harvesting effort. The two–dimensional state vector (y, z) represents biomasses and the two–dimensional control (v, w) is harvesting effort of each species, respectively. The catches are thus vy and wz (measured in biomass).[1]

The dynamics f in (1) is given by

$$f(y, z, v, w) = \begin{pmatrix} yR_y(y, z, v) \\ zR_z(y, z, w) \end{pmatrix} \quad \text{for all } (y, z, v, w) \in \mathbb{R}^4 , \qquad (12)$$

where $R_y : \mathbb{R}^3 \to \mathbb{R}$ and $R_z : \mathbb{R}^3 \to \mathbb{R}$ are two functions representing growth coefficients (the growth rates being $R_y - 1$ and $R_z - 1$). We do not make assumptions on the signs of R_y and R_z; the restrictions on the domain of variation of (y, z, v, w) will result from the requirement that the trajectories belong to the acceptable set $\mathbb{D}$, which will include signs considerations.

The discrete–time control system (1) with dynamics f given by (12) now writes:

$$\begin{cases} y(t+1) & = & y(t)R_y\big(y(t), z(t), v(t)\big) \\ z(t+1) & = & z(t)R_z\big(y(t), z(t), w(t)\big) . \end{cases} \qquad (13)$$

The implicit assumption in such functional form (12) for the dynamics is that, during one time period, the harvesting effort $v(t)$ of species y only affects the same species $y(t+1)$ and

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[1]In fact, any expression of the form $c(y, v)$ would fit for the catches in the following Proposition 1 as soon as $v \mapsto c(y, v)$ is strictly increasing and goes from 0 to $+\infty$ when v goes from 0 to $+\infty$. The same holds for $d(z, w)$ instead of wz.

not the other one $z(t+1)$ (the reverse holds for $w(t)$ and species z). Of course, after two periods, $z(t+2)$ depends on $y(t+1)$ which depends on $v(t)$ so that both efforts affect both species. Thus, the time period is assumed to be short enough for the impact on one species of harvesting the other species to be negligible.

The acceptable set $\mathbb{D}$ is defined by *minimal biomass thresholds* $y^\flat \geq 0$, $z^\flat \geq 0$ and *minimal catch thresholds* $Y^\flat \geq 0$, $Z^\flat \geq 0$:

$$\mathbb{D} = \{ (y, z, v, w) \in \mathbb{R}^4 \mid y \geq y^\flat, \ z \geq z^\flat, \ vy \geq Y^\flat, \ wz \geq Z^\flat \} . \tag{14}$$

Such a set includes both *preservation*

$$y(t) \geq y^\flat , \quad z(t) \geq z^\flat ,$$

and *production* requirements

$$v(t)y(t) \geq Y^\flat , \quad w(t)z(t) \geq Z^\flat$$

when thresholds are positive.

3.2 Expression of the viability kernel

The following Proposition 1 gives an explicit description of the viability kernel, under some conditions on the minimal thresholds.

Proposition 1 *Assume that the function* $R_y : \mathbb{R}^3 \to \mathbb{R}$ *is continuously decreasing*[2] *in the control* v *and satisfies* $\lim_{v \to +\infty} R_y(y, z, v) \leq 0$, *and that* $R_z : \mathbb{R}^3 \to \mathbb{R}$ *is continuously decreasing in the control variable* w, *and satisfies* $\lim_{w \to +\infty} R_z(y, z, w) \leq 0$. *If the thresholds in* (14) *are such that the following growth coefficients are greater than one*

$$R_y(y^\flat, z^\flat, \frac{Y^\flat}{y^\flat}) \geq 1 \ \text{ and } \ R_z(y^\flat, z^\flat, \frac{Z^\flat}{z^\flat}) \geq 1 , \tag{15}$$

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[2]In all that follows, a mapping $\varphi : \mathbb{R} \to \mathbb{R}$ is said to be increasing if $x \geq x' \Rightarrow \varphi(x) \geq \varphi(x')$. The reverse holds for decreasing. Thus, with this definition, a constant mapping is both increasing and decreasing.

the viability kernel associated with the dynamics f in (12) and the acceptable set $\mathbb{D}$ in (14) is given by

$$\mathbb{V}(f, \mathbb{D}) = \left\{ (y, z) \mid y \geq y^\flat, \ z \geq z^\flat, \ yR_y(y, z, \frac{Y^\flat}{y}) \geq y^\flat, \ zR_z(y, z, \frac{Z^\flat}{z}) \geq z^\flat \right\}. \qquad (16)$$

Before giving the proof, let us comment the assumptions. That the growth coefficients are decreasing with respect to the harvesting effort is a natural assumption. Conditions (15) mean that, at the point $(y^\flat, z^\flat)$ and applying efforts $u^\flat = \frac{Y^\flat}{y^\flat}$, $v^\flat = \frac{Z^\flat}{z^\flat}$, the growth coefficients are greater than one, hence both populations grow; hence, it could be thought that computing the viability kernel is useless since everything looks fine. However, if all is fine at the point $(y^\flat, z^\flat)$, it is not obvious that this also goes for a larger domain. Indeed, the ecosystem dynamics f given by (12) has no monotonocity properties that would allow to extend a result valid for a point to a whole domain. What is more, if continuous-time viability results mostly relies upon assumptions at the frontier of the constraints set, this is no longer true for discrete-time viability.

Proof. According to induction (9), we have:

$$\mathbb{V}_0 = \left\{ (y, z) \mid y \geq y^\flat, z \geq z^\flat \right\},$$

$$\mathbb{V}_1 = \left\{ (y, z) \ \middle| \ \begin{array}{c} y \geq y^\flat, z \geq z^\flat \text{ and, for some } (v, w) \geq 0, \\ vy \geq Y^\flat, wz \geq Z^\flat, yR_y(y, z, v) \geq y^\flat, zR_z(y, z, w) \geq z^\flat \end{array} \right\}$$

$$= \left\{ (y, z) \ \middle| \ y \geq y^\flat, z \geq z^\flat, yR_y(y, z, \frac{Y^\flat}{y}) \geq y^\flat, zR_z(y, z, \frac{Z^\flat}{z}) \geq z^\flat \right\}$$

because $v \mapsto R_y(y, z, v)$ and $w \mapsto R_z(y, z, w)$ are decreasing,

and thus we may select $v = \dfrac{Y^\flat}{y}$, $w = \dfrac{Z^\flat}{z}$.

Denoting $y' = yR_y(y, z, v)$, $z' = zR_z(y, z, w)$, we obtain,

$$\mathbb{V}_2 = \left\{ (y, z) \ \middle| \ \begin{array}{l} y \geq y^\flat, z \geq z^\flat \text{ and, for some } (v, w) \geq 0, \\ vy \geq Y^\flat, \ wz \geq Z^\flat \\ y' \geq y^\flat, \ y'R_y(y', z', \frac{Y^\flat}{y'}) \geq y^\flat, \ z' \geq z^\flat, \ z'R_z(y', z', \frac{Z^\flat}{z'}) \geq z^\flat \end{array} \right\}.$$

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We shall now make use of the property, recalled in Sect. 2, that when the decreasing sequence $(\mathbb{V}_k)_{k \in \mathbb{N}}$ of viability kernels up to time k is stationary, its limit is the viability kernel $\mathbb{V}(f, \mathbb{D})$. Hence, it suffices to show that $\mathbb{V}_1 \subset \mathbb{V}_2$ to obtain that $\mathbb{V}(f, \mathbb{D}) = \mathbb{V}_1$. Let $(y, z) \in \mathbb{V}_1$, so that

$$y \geq y^\flat, \quad z \geq z^\flat \text{ and } yR_y(y, z, \frac{Y^\flat}{y}) \geq y^\flat, \quad zR_z(y, z, \frac{Z^\flat}{z}) \geq z^\flat .$$

Since $R_y : \mathbb{R}^3 \to \mathbb{R}$ is continuously decreasing in the control variable, with $\lim_{v \to +\infty} R_y(y, z, v) \leq 0$, and since $yR_y(y, z, \frac{Y^\flat}{y}) \geq y^\flat$, there exists a $\hat{v} \geq \frac{Y^\flat}{y}$ (depending on y and z) such that $y' = yR_y(y, z, \hat{v}) = y^\flat$. The same holds for $R_z : \mathbb{R}^3 \to \mathbb{R}$ and $z' = zR_z(y, z, \hat{w}) = z^\flat$. By (15), we deduce that

$$y'R_y(y', z', \frac{Y^\flat}{y'}) = y^\flat R_y(y^\flat, z^\flat, \frac{Y^\flat}{y^\flat}) \geq y^\flat \text{ and } z'R_z(y', z', \frac{Z^\flat}{z'}) = z^\flat R_z(y^\flat, z^\flat, \frac{Z^\flat}{z^\flat}) \geq z^\flat .$$

The inclusion $\mathbb{V}_1 \subset \mathbb{V}_2$ follows. $\square$

As a direct consequence of the proof of Proposition 1, for each $(y, z) \in \mathbb{V}(f, \mathbb{D})$, the control $(\hat{v}, \hat{w})$ belongs to $\mathbb{U}_{\mathbb{V}(f,\mathbb{D})}(y, z)$, the set of viable controls defined in (11). More explicitly, we have the following expression of the viable controls set, which results from the above observation and proof.

Corollary 2 *Suppose that the assumptions of Proposition 1 are satisfied. Denoting*

$$\begin{cases} \hat{v}(y, z) & = \max\{v \geq \frac{Y^\flat}{y} \mid yR_y(y, z, v) = y^\flat\} \\ \hat{w}(y, z) & = \max\{w \geq \frac{Z^\flat}{z} \mid zR_z(y, z, w) = z^\flat\} , \end{cases}$$

the set of viable controls is given by

$$\mathbb{U}_{\mathbb{V}(f,\mathbb{D})}(y, z) = \left\{ (v, w) \left| \begin{array}{l} \hat{v}(y, z) \geq v \geq \frac{Y^\flat}{y}, \quad \hat{w}(y, z) \geq w \geq \frac{Z^\flat}{z}, \\ y'R_y(y', z', \frac{Y^\flat}{y'}) \geq y^\flat, \ z'R_z(y', z', \frac{Z^\flat}{z'}) \geq z^\flat \end{array} \right. \right\}$$

where $y' = yR_y(y, z, v)$, $z' = zR_z(y, z, w)$.

3.3 Viability kernel for a nonlinear predator–prey system

Till now, no trophic relationship has been specified between species y and z. From now on, we shall focus on nonlinear predator–prey systems. The two–dimensional state vector (y, z) represents biomasses of preys (y) and predators (z). This interpretation results from the monotonicity assumptions made on function R_z in the following Proposition 3, which provides a simpler version of Proposition 1 when the predator z does not exhibit density–dependence.

Proposition 3 *Suppose that the dynamics f is given by*

$$f(y, z, v, w) = \begin{pmatrix} yR_y(y, z, v) \\ zR_z(y, w) \end{pmatrix} \quad for\ all\ (y, z, v, w) \in \mathbb{R}^4 . \tag{17}$$

Assume that the function $R_y : \mathbb{R}^3 \to \mathbb{R}$ is decreasing in the control v and $R_z : \mathbb{R}^2 \to \mathbb{R}$ is increasing in the state variable y and continuously decreasing in the control variable w, and satisfies $\lim_{w \to +\infty} R_z(y, w) \leq 0$. If

$$R_y(y^\flat, z^\flat, \frac{Y^\flat}{y^\flat}) \geq 1 \ \ and \ \ R_z(y^\flat, \frac{Z^\flat}{z^\flat}) \geq 1 , \tag{18}$$

the viability kernel associated with the dynamics f in (17) and the acceptable set $\mathbb{D}$ in (14) is given by

$$\mathbb{V}(f, \mathbb{D}) = \left\{ (y, z) \mid y \geq y^\flat,\ z \geq z^\flat,\ yR_y(y, z, \frac{Y^\flat}{y}) \geq y^\flat \right\} . \tag{19}$$

Proof. The proof follows the same lines as the one of Proposition 1. The description of the viability kernel is simpler because of the following argument. The monotonicity assumptions on each component of function R_z and condition $R_z(y^\flat, \frac{Z^\flat}{z^\flat}) \geq 1$ lead to

$$y \geq y^\flat \ and \ z \geq z^\flat \Rightarrow R_z(y, \frac{Z^\flat}{z}) \geq R_z(y, \frac{Z^\flat}{z^\flat}) \geq R_z(y^\flat, \frac{Z^\flat}{z^\flat}) \geq 1 = \frac{z^\flat}{z^\flat} \geq \frac{z^\flat}{z} .$$

Therefore, $zR_z(y, \frac{Z^\flat}{z}) \geq z^\flat$ for all $y \geq y^\flat$ and $z \geq z^\flat$. This explains why (16) reduces to (19). □

3.4 Viability kernel for a Lotka–Volterra system

Consider the following discrete–time Lotka–Volterra system of equations with density–dependence in the prey

$$\begin{cases} y(t+1) & = & Ry(t) - \dfrac{R}{\kappa}y^2(t) - \alpha y(t)z(t) - v(t)y(t) \,, \\[2mm] z(t+1) & = & Lz(t) + \beta y(t)z(t) - w(t)z(t) \,, \end{cases} \tag{20}$$

where $R > 1$, $0 < L < 1$, $\alpha > 0$, $\beta > 0$ and $\kappa = \frac{R}{R-1}K$, with $K > 0$ the carrying capacity for prey. The dynamics f is as in (17), with $R_y(y, z, v) = R - \frac{R}{\kappa}y - \alpha z - v$ and $R_z(y, w) = L + \beta y - w$. Proposition 3 gives the following Corollary.

Corollary 4 *Consider the Lotka–Volterra predator–prey model* (20). *Whenever*

$$y^\flat \geq \frac{1 - L}{\beta} \quad \text{and} \quad z^\flat \leq \frac{R - 1}{\alpha} - \frac{R(1 - L)}{\alpha\beta\kappa} \,,$$

any minimal catch thresholds $Y^\flat$ and $Z^\flat$ such that

$$Y^\flat \leq Y^{\flat,\star} := y^\flat(R - \tfrac{R}{\kappa}y^\flat - \alpha z^\flat - 1) \tag{21a}$$

$$Z^\flat \leq Z^{\flat,\star} := z^\flat(L + \beta y^\flat - 1) \,, \tag{21b}$$

satisfy (18) *and the viability kernel associated with the dynamics f in* (20) *and the acceptable set $\mathbb{D}$ in* (14) *is given by*

$$\mathbb{V}(f, \mathbb{D}) = \left\{ (y, z) \,\Big|\, y \geq y^\flat, \ z^\flat \leq z \leq \frac{1}{\alpha}\left[R\left(\frac{\kappa - y}{\kappa}\right) - \frac{Y^\flat + y^\flat}{y} \right] \right\} \,. \tag{22}$$

This Corollary has the following practical consequence. An initial state (y, z) such that $y \geq y^\flat$ and $z \geq z^\flat$ belongs to all the viability kernels $\mathbb{V}(f, \mathbb{D})$ associated to $Z^\flat \leq Z^{\flat,\star}$, by (21b), and to $Y^\flat \leq \min\left\{Y^{\flat,\star}, y(R - Ry/\kappa - \alpha z) - y^\flat\right\}$, by (21a) and (22), as long as this last quantity is nonnegative. In other words, if viably managed, the fishery could produce at least $\min\left\{Y^{\flat,\star}, y(R - Ry/\kappa - \alpha z) - y^\flat\right\}$ and $Z^{\flat,\star}$, while respecting biological thresholds $y^\flat$ and $z^\flat$. We shall use this property in the following numerical application.

4 Numerical application to the hake–anchovy couple in the Peruvian upwelling ecosystem

We provide a viability analysis of the hake–anchovy Peruvian fisheries between the years 1971 and 1981. For this, we shall consider a discrete-time Lotka–Volterra model for the couple anchovy (prey y) and hake (predator z), for which the viability kernel has explicit description. The emphasis is not on developing a biological model, but rather on decision-making using such a model.

The period between the years 1971 and 1981 has been chosen because the competition between the fishery and hake was reduced due to low anchovy catches, and because of the absence a strong warm event after El Niño 1972. We have 11 couples of biomasses, and the same for catches. The 5 parameters of the model are estimated minimizing a weighted residual squares sum function using a conjugate gradient method, with central derivatives. Estimated parameters and comparisons of observed and simulated biomasses are shown in Figure 1.

We consider values of $y^\flat = 7\,000\,000$ t and $z^\flat = 200\,000$ t for minimal biomass thresholds and values of $Y^\flat = 2\,000\,000$ t and $Z^\flat = 5\,000$ t for minimal catch thresholds (IMARPE, 2000, 2004). Conditions (18) in Proposition 3 are satisfied with these values. Indeed, the expressions in (21a)–(21b) give:

$$
\begin{cases}
Y^\flat &=& 2\,000\,000 \text{ t} &\leq& Y^{\flat,\star} &=& 5\,399\,000 \text{ t} \\
Z^\flat &=& 5\,000 \text{ t} && \leq & Z^{\flat,\star} &=& 56\,800 \text{ t}\,.
\end{cases}
\tag{23}
$$

The viability kernel is depicted in Figure 2. The unique viable point (within the viability kernel) is the initial point. Thus, based upon this model, the fishery could have been managed – with appropriate viable controls – to produce catches above $Y^\flat = 2\,000\,000$ t and $Z^\flat = 5\,000$ t, while ensuring biological conservation. What is more, due to the remark following Corollary 4, captures up to $Y^\flat = 5\,399\,000$ t and $Z^\flat = 56\,800$ t were theoreticaly achievable in a sustainable way starting from year 1971.
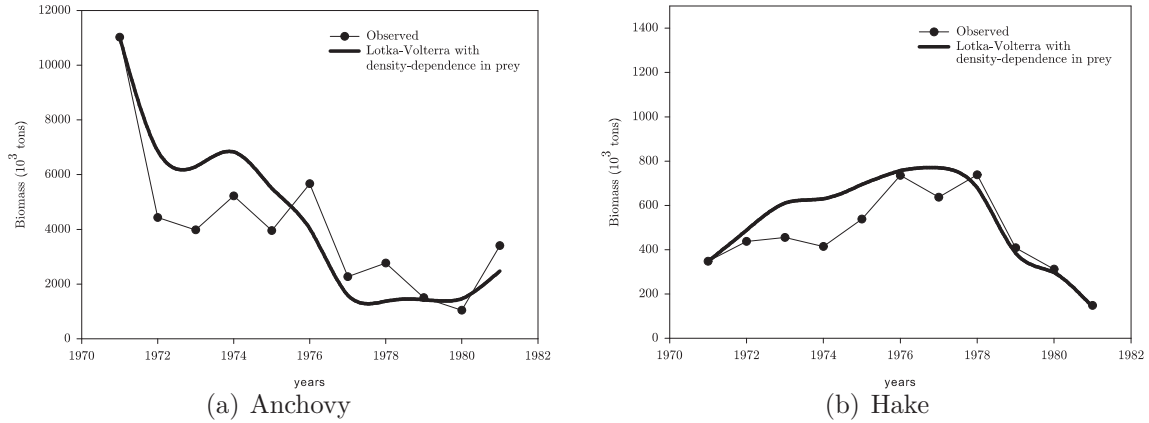
(a) Anchovy (b) Hake

Figure 1: Comparison of observed and simulated biomasses of anchovy and hake using a Lotka–Volterra model with density-dependence in the prey. Model parameters are $R = 2.25 \text{ year}^{-1}$, $L = 0.945 \text{ year}^{-1}$, $\kappa = 67\,113 \times 10^3 \text{ t}$ ($K = 37\,285 \times 10^3 \text{ t}$), $\alpha = 1.220 \times 10^{-6} \text{ t}^{-1}$, $\beta = 4.845 \times 10^{-8} \text{ t}^{-1}$.

5 Conclusion

Motivated by viable management of ecosystems, we have provided a general condition ensuring an explicit construction of viability kernels and have applied this to the viability analysis of generic ecosystem models with harvesting.

Our results have then been applied to a Lotka–Volterra model using the anchovy–hake couple in the Peruvian upwelling ecosystem. We showed that, during the anchovy collapse, theoretically the fishery could have been viably managed to produce catches above the expected threshold levels while ensuring biological conservation.

It is interesting to notice that the kind of maximum sustainable yields $Y^{\flat,\star} = 5\,399\,000 \text{ t}$ and $Z^{\flat,\star} = 56\,800 \text{ t}$ provided by our approach in (23) are comparable to the $4\,250\,000 \text{ t}$ anchovy yield and the $55\,000 \text{ t}$ hake yield, respectively, established for the year 2006 (PRODUCE, 2005, 2006b,c), or to the $5\,000\,000 \text{ t}$ anchovy yield and the $35\,000 \text{ t}$ hake yield, respectively, established for the year 2007 (PRODUCE, 2006a, 2007a,b). So, despite simplicity[3] of the models considered, our approach may provide a mean of designing sustain-

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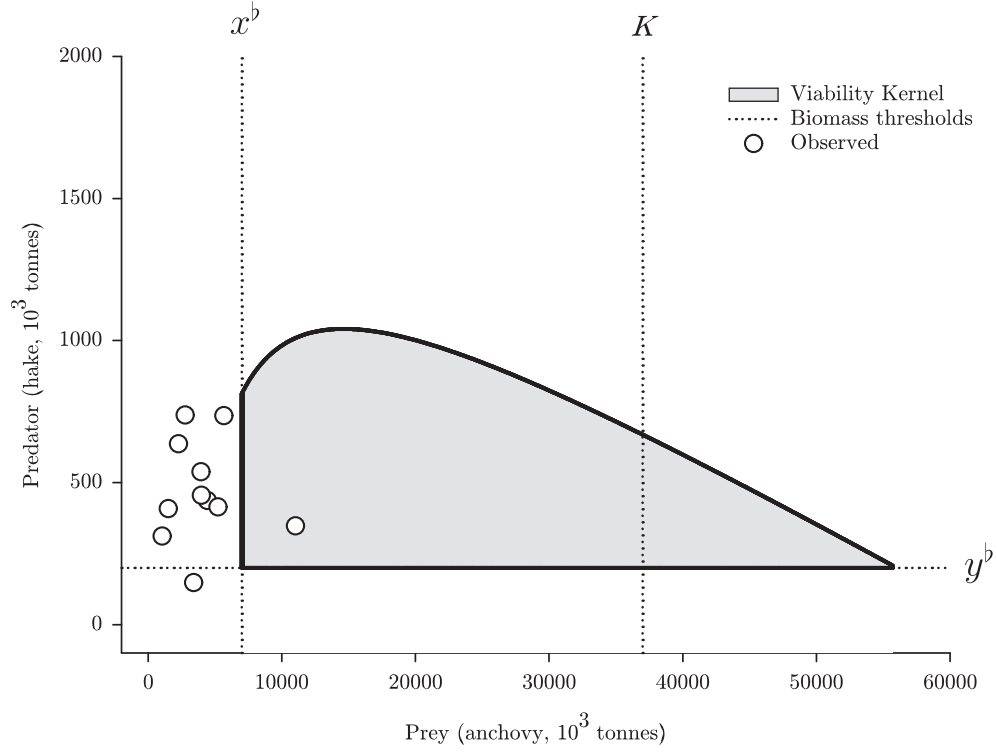[3]In addition to hake, there are other important predators of anchovy in the Peruvian upwelling ecosystem,

Figure 2: Viability kernel (in grey) for a Lotka–Volterra model with density-dependence in the prey in the predator–prey phase space (with $y^\flat = 7\,000\,000$ t, $z^\flat = 200\,000$ t, $Y^\flat = 2\,000\,000$ t, $Z^\flat = 5\,000$ t). The unique point within the viability kernel is the initial point, while all subsequent points in the trajectory are outside the state constraint set.

able yields from an ecosystem point of view.

Thus, control and viability theory methods have allowed us to introduce ecosystem considerations, such as multispecies and multiobjectives, and have contributed to integrate the long term dynamics, which is generally not considered in conventional fishery management. Notice that the World Summit on Sustainable Development (Garcia, Zerbi, Aliaume, Chi, and Lasserre, 2003) encouraged the application of the ecosystem approach by 2010.

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such as mackerel and horse mackerel, seabirds and pinnipeds, which were not considered. Also, anchovy has been an important prey of hake, but other prey species have been found in the opportunistic diet of hake (Tam, Purca, Duarte, Blaskovic, and Espinoza, 2006)

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